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The classical supersymmetric Coulomb problem

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Abstract

After setting up a general model for supersymmetric classical mechanics in more than one dimension we describe systems with centrally symmetric potentials and their Poisson algebra. We then apply this information to the investigation and solution of the supersymmetric Coulomb problem, specified by an $\frac{1}{|\boldsymbol{x}|}$ repulsive bosonic potential.

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1. Introduction

The introduction of supersymmetry into classical mechanics has been motivated in part by the study of supersymmetric *quantum* mechanics, e.g. in the influential paper by Witten [1], not the least with the aim to find a classical 'background' for these models.

That this can be achieved has been long known, namely that classical supersymmetric mechanics can be consistently constructed by replacing all dynamical quantities of the theory in question by even and odd elements of a Grassmann algebra $\mathcal{B}$ according to their respective bosonic or fermionic nature. This Grassmann-valued mechanics has been studied first by Berezin and Marinov [2] and Casalbuoni [3]. A more recent account devoted to one-dimensional systems was given by Junker and Matthiesen in [4]. For a general review of Grassmannian geometry the reader may turn to the book by de Witt [5].

An interesting but somewhat different approach has been taken in a series of papers by Gozzi *et al* [6]. Using path integral methods in classical mechanics he and co-authors construct an $N = 2$ supersymmetric theory for any given Hamiltonian system, where fermionic fields correspond to sections of various cotangent and tangent bundles of the configuration manifold. It should be noted, however, that the resulting model differs from the one analysed in this paper, so that solutions or symmetries cannot be directly compared.

While there is thus a considerable amount of literature on supersymmetric mechanics and interest in the topic has grown recently, there is a surprisingly low number of specific models that have been investigated and solved. Furthermore, attention is usually restricted to the case of one spatial dimension. An interesting exception on two-dimensional Liouville systems is given in [7].

We try in this paper to address both of these shortcomings, first by establishing some general properties of multidimensional supersymmetric mechanics, especially for centrally symmetric potentials and then by analysing and solving a particular classical model, the Coulomb problem. The $1/|\boldsymbol{x}|$-potential is one of the oldest and most prominent potentials that has been studied in physics. This is not only because it properly describes both the gravitational and the electrical interactions of massive or charged point particles, respectively, but also because it has some very attractive features. One example is that by a theorem of Bertrand [8] the attractive version of the potential is the only centrally symmetric potential next to the harmonic oscillator such that every admissable trajectory is closed. Furthermore, one can derive an explicit time-dependent solution for every given set of initial data. This is a very useful property because it implies, if perhaps unexpectedly at first sight, that the supersymmetric version of the theory can be explicitly solved, too.

This paper is based on the experience gained from an earlier investigation into one-dimensional supersymmetric problems. Both in [9] and [10], where a different concept of complex conjugation for Grassmann variables was used, one of the ideas of tackling the supersymmetric problem was the assumption of a *finitely* generated Grassmann algebra $\mathcal{B}$. This led to a layer-by-layer structure of the theory which could be used to solve one-dimensional models by subsequently solving higher layers based on the solutions of all lower ones. We shall take a similar approach in this paper.

The starting point for our investigation will be the statement of the Lagrangian in section 2. It is obtained ultimately from the familiar $N = 2$ supersymmetric $(1 + 1)$-dimensional field theory with Yukawa interaction via dimensional reduction which leads to the description of one-dimensional supersymmetric point particle mechanics analysed in [9]. The resulting Lagrangian can then be generalized to more than one dimension. Note that supersymmetry requires the bosonic potential to be non-negative. This means that our attention has to be restricted to the case of repulsive interaction.

We go on to derive the complete set of $3n$ equations of motion and show that as in the one-dimensional model there are two independent supersymmetries as well as a purely fermionic invariance transformation which can be seen as an internal 'rotation' of the fermionic variables. We proceed by giving one explicit solution to the fermionic equations of motion utilizing the supercharges—except in the one-dimensional case this cannot be the complete solution though. We then restrict our attention to a special class of models, namely those with a centrally symmetric scalar potential. After restating the equations of motion we derive in section 3 the explicit form of the supersymmetric angular momentum and show that it is a supersymmetric invariant.

The symmetries of our model can be understood best from its Poisson algebra which we therefore describe completely in section 4.

In section 5, we write the complete set of equations governing the supersymmetric Coulomb problem. Even in three dimensions these are still nine coupled first- and second-order differential equations and it seems by no means obvious that there will be an explicit analytical solution for every possible set of initial data.

We use section 6 to show that such an explicit solution exists. This is achieved by utilizing a method originally employed in [10] and then applied further in [9]. The key idea is to decompose all dynamical quantities (and therefore every equation) into terms containing the same number of Grassmann generators. The model then naturally obtains a layer structure which can be analysed from the bottom layer, the classical problem, upwards. In particular, solutions to higher layers may be found by using all those already derived for the lower layers.

We will carry out this programme here up to second order in the Grassmann algebra—since our results indicate a close resemblance of the multi-dimensional to the one-dimensional case we can suspect that higher-order solutions exhibit many of the characteristics described already for the one-dimensional case. We will give a short summary of these in our conclusions.

As mentioned we begin our analysis in section 6.1 with the bottom layer of the system which is identical to the classical, i.e. non-supersymmetrized problem. Although the solutions to the problem are well known we explain briefly how explicit time-dependent expressions can be found—we do this because the textbook treatments of the problem usually only derive the possible orbits but do not specify the actual functions of time that describe it.

The fermionic equations of motion comprise the next layer and we study their solutions in detail in subsection 6.2. As in the one-dimensional case we find that fermionic motion is not only compact but also restricted to a spherical space. Furthermore, while there is a plane of motion for the bosonic variables (of lowest order) the same is not true for the fermionic variables or, in other words, fermionic motion is non-trivial in all three spatial directions.

Finally, we solve the top layer of our system, consisting of the bosonic quantities of second order in the generators. Though the particular solutions found in the one-dimensional case cannot be applied directly to the Coulomb problem we can carry over the most important aspect to the three-dimensional case, namely that solutions to the homogeneous equations of motion of the top layer system are *variations* of the bottom layer system with the free parameters of that motion. In particular, we show that every top layer solution to the homogeneous equations of the Coulomb problem can be explained as a linear combination of the variations with initial time, energy, eccentricity, orientation of angular momentum and orientation of the hyperbola in the plane of motion. We conclude our analysis by a full statement of the explicit solutions of the inhomgeneous equations including the boson–fermion interaction term.

As is well known, the classical Coulomb problem exhibits a hidden $O(3,1)$-symmetry as a consequence of which there is an extra vector-valued conserved quantity, the Runge–Lenz vector. It comes as an unexpected surprise that this symmetry is broken in the supersymmetric version of the problem. We demonstrate this, i.e. that there can be no supersymmetric version of the Runge–Lenz vector, in the appendix.

2. Multidimensional supersymmetric mechanics

We assume that supersymmetric mechanical models can be described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} - \frac{1}{2}\boldsymbol{U}(\boldsymbol{x}) \cdot \boldsymbol{U}(\boldsymbol{x}) + \frac{\mathrm{i}}{2}\boldsymbol{\psi}_+ \cdot \dot{\boldsymbol{\psi}}_+ + \frac{\mathrm{i}}{2}\boldsymbol{\psi}_- \cdot \dot{\boldsymbol{\psi}}_- + \mathrm{i}\boldsymbol{\psi}_+ \nabla\boldsymbol{U}(\boldsymbol{x})\boldsymbol{\psi}_- \tag{1}$$

which is a generalization of the familiar supersymmetric Lagrangian in one-dimensional mechanics [9], ultimately derived from $(1+1)$-dimensional field theory. The dynamical quantities $\boldsymbol{x}$, $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ are n-component vectors, e.g. $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{x}$ is Grassmann-even and $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ are Grassmann-odd.

In addition, $\boldsymbol{U}$ is taken to be a vector-valued n-component Grassmann-even function of $\boldsymbol{x}$, so that $\nabla\boldsymbol{U}$ is an $n \times n$-tensor with elements $\partial_i U_j(x_k)$. We will usually take this tensor to be symmetric which will always be the case if $\boldsymbol{U}$ is derived from a scalar potential term as $\boldsymbol{U}(\boldsymbol{x}) = \nabla W(\boldsymbol{x})$.

$\boldsymbol{x} \cdot \boldsymbol{y}$ denotes the standard Euclidean inner product, extended if necessary in the obvious way: $\boldsymbol{x}\boldsymbol{M}\boldsymbol{y}$, $\boldsymbol{M}$ being an $n \times n$-tensor, stands, e.g. for the component expression $x^i M_{ij} y^j$.

The equations of motion can now be read off from the Lagrangian:

$$\ddot{\boldsymbol{x}} = -\nabla\boldsymbol{U}(\boldsymbol{x})\boldsymbol{U}(\boldsymbol{x}) + \mathrm{i}\nabla(\boldsymbol{\psi}_+ \nabla\boldsymbol{U}(\boldsymbol{x})\boldsymbol{\psi}_-) \tag{2}$$

$$\dot{\boldsymbol{\psi}}_+ = -\nabla \boldsymbol{U}(\boldsymbol{x})\boldsymbol{\psi}_- \tag{3}$$

$$\dot{\boldsymbol{\psi}}_- = \nabla \boldsymbol{U}(\boldsymbol{x})\boldsymbol{\psi}_+. \tag{4}$$

Every equation is vector-valued and has n components.

Invariance with respect to time-translation means that there is a conserved Hamiltonian which can be calculated to be

$$H = \tfrac{1}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} + \tfrac{1}{2}\boldsymbol{U}(\boldsymbol{x})\cdot\boldsymbol{U}(\boldsymbol{x}) - \mathrm{i}\boldsymbol{\psi}_+\nabla\boldsymbol{U}(\boldsymbol{x})\boldsymbol{\psi}_-. \tag{5}$$

In addition, there are two independent supersymmetry transformations, namely,

$$\begin{array}{lll} \delta\boldsymbol{x} = \mathrm{i}\epsilon\boldsymbol{\psi}_+ & \delta\boldsymbol{\psi}_+ = -\epsilon\dot{\boldsymbol{x}} & \delta\boldsymbol{\psi}_- = -\epsilon\boldsymbol{U}(\boldsymbol{x}) \\ \tilde{\delta}\boldsymbol{x} = \mathrm{i}\epsilon\boldsymbol{\psi}_- & \tilde{\delta}\boldsymbol{\psi}_+ = \epsilon\boldsymbol{U}(\boldsymbol{x}) & \tilde{\delta}\boldsymbol{\psi}_- = -\epsilon\dot{\boldsymbol{x}} \end{array} \tag{6}$$

where ϵ is an infinitesimal scalar Grassmann-odd parameter. The corresponding conserved supercharges are given by the expressions

$$Q = \dot{\boldsymbol{x}}\cdot\boldsymbol{\psi}_+ + \boldsymbol{U}(\boldsymbol{x})\cdot\boldsymbol{\psi}_- \tag{7}$$

$$\tilde{Q} = \dot{\boldsymbol{x}}\cdot\boldsymbol{\psi}_- - \boldsymbol{U}(\boldsymbol{x})\cdot\boldsymbol{\psi}_+. \tag{8}$$

Finally, an internal transformation in the fermionic part of the Grassmann algebra, given by

$$\delta\boldsymbol{\psi}_+ = \eta\boldsymbol{\psi}_- \qquad \delta\boldsymbol{\psi}_- = -\eta\boldsymbol{\psi}_+$$

where η denotes an infinitesimal scalar Grassmann-even parameter, also leaves the Lagrangian invariant. The resulting charge is simply

$$R = \mathrm{i}\boldsymbol{\psi}_+\cdot\boldsymbol{\psi}_-. \tag{9}$$

One of the results from [9], which can be carried over from the one-dimensional case almost without alteration, is that solutions to the fermionic equations can be found using the supercharges. In one dimension (7) and (8) can be formally inverted to yield

$$\boldsymbol{\psi}_+ = \frac{1}{2E}(Q\dot{\boldsymbol{x}} - \tilde{Q}\boldsymbol{U}) \tag{10}$$

$$\boldsymbol{\psi}_- = \frac{1}{2E}(Q\boldsymbol{U} + \tilde{Q}\dot{\boldsymbol{x}}) \tag{11}$$

where E is the constant Grassmann-valued energy. This is still a solution in the n-dimensional case, though special care has to be taken if the real part E_0 is zero.

Unlike the situation in one dimension, however, (10) and (11) give only part of the fermionic solution. In the multi-dimensional case there is still a $2(n-1)$-dimensional solution space that needs to be determined.

3. Centrally symmetric potentials and angular momentum

So far these formulae have been straightforward generalizations of formulae from the one-dimensional case. However, in more than one dimension there can be additional conserved quantities if there are extra symmetries in the system. We will now assume that the scalar bosonic potential

$$V = \tfrac{1}{2}\boldsymbol{U}(\boldsymbol{x})\cdot\boldsymbol{U}(\boldsymbol{x})$$

is central, i.e. a function of $|\boldsymbol{x}| = (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}$ only. This can be easily achieved by demanding that the scalar potential $W(\boldsymbol{x})$ is a function of $|\boldsymbol{x}|$, for then $\boldsymbol{U}(\boldsymbol{x})$ will be of the form

$$\boldsymbol{U}(\boldsymbol{x}) = W'(|\boldsymbol{x}|)\hat{\boldsymbol{x}}$$

with $\hat{\boldsymbol{x}}$ denoting the unit vector $\boldsymbol{x}/|\boldsymbol{x}|$.

The equations of motion can now be written as follows:

$$\ddot{\boldsymbol{x}} = -W'W''\hat{\boldsymbol{x}} + \mathrm{i}\left(\frac{W''|\boldsymbol{x}| - W'}{|\boldsymbol{x}|^2}\right)(\boldsymbol{\psi}_+(\boldsymbol{\psi}_- \cdot \hat{\boldsymbol{x}}) + (\boldsymbol{\psi}_+ \cdot \hat{\boldsymbol{x}})\boldsymbol{\psi}_-)$$
$$+ \mathrm{i}\left[\left(\frac{W'''|\boldsymbol{x}|^2 - 3(W''|\boldsymbol{x}| - W')}{|\boldsymbol{x}|^2}\right)(\boldsymbol{\psi}_+ \cdot \hat{\boldsymbol{x}})(\boldsymbol{\psi}_- \cdot \hat{\boldsymbol{x}}) + \left(\frac{W''|\boldsymbol{x}| - W'}{|\boldsymbol{x}|^2}\right)(\boldsymbol{\psi}_+ \cdot \boldsymbol{\psi}_-)\right]\hat{\boldsymbol{x}} \tag{12}$$

$$\dot{\boldsymbol{\psi}}_+ = -\left(\frac{W'}{|\boldsymbol{x}|}\right)\boldsymbol{\psi}_- - \left(\frac{W''|\boldsymbol{x}| - W'}{|\boldsymbol{x}|}\right)(\boldsymbol{\psi}_- \cdot \hat{\boldsymbol{x}})\hat{\boldsymbol{x}} \tag{13}$$

$$\dot{\boldsymbol{\psi}}_- = \left(\frac{W'}{|\boldsymbol{x}|}\right)\boldsymbol{\psi}_+ + \left(\frac{W''|\boldsymbol{x}| - W'}{|\boldsymbol{x}|}\right)(\boldsymbol{\psi}_+ \cdot \hat{\boldsymbol{x}})\hat{\boldsymbol{x}} \tag{14}$$

where the argument of W is $|\boldsymbol{x}|$. Note that the purely bosonic part of the problem is described just by the first term on the right-hand side of (12).

For convenience we will now restrict ourselves to the three-dimensional case relevant for the Coulomb problem, although our discussion should generalize easily to more than three dimensions.

The particular form of the potential function W means that there is an extra symmetry in the system, given by rotating both the bosonic and the fermionic variables by the same amount around the same axis $\boldsymbol{a}$. Infinitesimally, we can write

$$\delta_a \boldsymbol{x} = \epsilon[\boldsymbol{a}, \boldsymbol{x}] \qquad \delta_a \boldsymbol{\psi}_+ = \epsilon[\boldsymbol{a}, \boldsymbol{\psi}_+] \qquad \delta_a \boldsymbol{\psi}_- = \epsilon[\boldsymbol{a}, \boldsymbol{\psi}_-]$$

where $[\cdot, \cdot]$ denotes the three-dimensional vector product, i.e. $[\boldsymbol{x}, \boldsymbol{y}]^i = \epsilon_{ijk} x_j y_k$. This leads to the conserved angular momentum

$$\boldsymbol{L} = [\boldsymbol{x}, \dot{\boldsymbol{x}}] - \frac{\mathrm{i}}{2}[\boldsymbol{\psi}_+, \boldsymbol{\psi}_+] - \frac{\mathrm{i}}{2}[\boldsymbol{\psi}_-, \boldsymbol{\psi}_-]. \tag{15}$$

Note that the second and third terms of $\boldsymbol{L}$ are non-zero since $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ are Grassmann-odd. It is an interesting observation that the strict condition for the existence of angular momentum is that W is central. By choosing $\boldsymbol{U} = f(|\boldsymbol{x}|)\boldsymbol{O}\boldsymbol{x}$; $\boldsymbol{O} \in O(3)$ we end up with a rotationally invariant bosonic potential V but angular momentum is not conserved.

Finally, we point out that the following two expressions are also time-independent:

$$[\boldsymbol{\psi}_+, \boldsymbol{\psi}_+] \cdot [\boldsymbol{\psi}_-, \boldsymbol{\psi}_-]$$
$$([\boldsymbol{\psi}_+, \boldsymbol{\psi}_+] \cdot \boldsymbol{\psi}_+) \cdot ([\boldsymbol{\psi}_-, \boldsymbol{\psi}_-] \cdot \boldsymbol{\psi}_-)$$

which can be said to denote the product of the oriented 'areas' and 'volumes' of $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$, respectively. They do not, however, constitute independently conserved quantities as the first one is proportional to R^2 and the second one to R^3.

Apart from invariance under time translation, another important aspect is that of invariance under supersymmetry. It can be shown directly that angular momentum and energy are supersymmetric invariants whereas, e.g. the extra charge R is not:

$$\begin{array}{lllll} \delta H = 0 & \delta \boldsymbol{L} = \boldsymbol{0} & \delta Q = -2\epsilon H & \delta\tilde{Q} = 0 & \delta R = -\mathrm{i}\epsilon\tilde{Q} \\ \tilde{\delta} H = 0 & \tilde{\delta}\boldsymbol{L} = \boldsymbol{0} & \tilde{\delta} Q = 0 & \tilde{\delta}\tilde{Q} = -2\epsilon H & \tilde{\delta} R = \mathrm{i}\epsilon Q. \end{array} \tag{16}$$

Here δ and $\tilde{\delta}$ are the two supersymmetry transformations introduced in equation (6) and ϵ is a Grassmann-odd infinitesimal parameter.

4. Poisson-algebra

For a better understanding of the symmetries of our mechanical system we will now derive the Poisson-algebra of the problem. Therefore, we must write the definition of the Poisson brackets for Grassmann-valued quantities. To do this we have to introduce the canonical momenta $\boldsymbol{p}$ and $\boldsymbol{\pi}$ by

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} \qquad \boldsymbol{\pi}_+ = \frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_+} \qquad \boldsymbol{\pi}_- = \frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_-}.$$

For the specific choice of our model one finds immediately that

$$\boldsymbol{p} = \dot{\boldsymbol{x}} \qquad \boldsymbol{\pi}_+ = -\frac{\mathrm{i}}{2}\boldsymbol{\psi}_+ \qquad \boldsymbol{\pi}_- = -\frac{\mathrm{i}}{2}\boldsymbol{\psi}_-. \tag{17}$$

The fact that the canonical momenta associated with the fermionic variables are up to a constant factor identical to these variables themselves means that $\dot{\boldsymbol{\psi}}_+$ and $\dot{\boldsymbol{\psi}}_-$ cannot be expressed uniquely as a function of configuration and momentum variables separately. It is therefore sensible to view (17) as a constraint and replace all occurrences of $\boldsymbol{\pi}_+$ and $\boldsymbol{\pi}_-$ accordingly. Hamilton equations can then be written as

$$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{x}} \qquad \dot{\boldsymbol{\psi}}_+ = -\mathrm{i}\frac{\partial H}{\partial \boldsymbol{\psi}_+} \qquad \dot{\boldsymbol{\psi}}_- = -\mathrm{i}\frac{\partial H}{\partial \boldsymbol{\psi}_-}$$

so that the Poisson brackets for even quantities y and odd quantities θ have to be

$$\{y_1, y_2\} = \left(\frac{\partial y_1}{\partial x_i}\frac{\partial y_2}{\partial p_i} - \frac{\partial y_2}{\partial x_i}\frac{\partial y_1}{\partial p_i} + \mathrm{i}\frac{\partial y_1}{\partial \psi_{+i}}\frac{\partial y_2}{\partial \psi_{+i}} + \mathrm{i}\frac{\partial y_1}{\partial \psi_{-i}}\frac{\partial y_2}{\partial \psi_{-i}}\right)$$

$$\{\theta, y\} = \left(\frac{\partial \theta}{\partial x_i}\frac{\partial y}{\partial p_i} - \frac{\partial y}{\partial x_i}\frac{\partial \theta}{\partial p_i} - \mathrm{i}\frac{\partial \theta}{\partial \psi_{+i}}\frac{\partial y}{\partial \psi_{+i}} - \mathrm{i}\frac{\partial \theta}{\partial \psi_{-i}}\frac{\partial y}{\partial \psi_{-i}}\right)$$

$$\{\theta_1, \theta_2\} = \left(\frac{\partial \theta_1}{\partial x_i}\frac{\partial \theta_2}{\partial p_i} + \frac{\partial \theta_2}{\partial x_i}\frac{\partial \theta_1}{\partial p_i} - \mathrm{i}\frac{\partial \theta_1}{\partial \psi_{+i}}\frac{\partial \theta_2}{\partial \psi_{+i}} - \mathrm{i}\frac{\partial \theta_1}{\partial \psi_{-i}}\frac{\partial \theta_2}{\partial \psi_{-i}}\right)$$

where summation over i is implied. The result is the same as in [3] if the constraint (17) is used to replace the occurrences of $\boldsymbol{\pi}_+$ and $\boldsymbol{\pi}_-$ in the formulae given there.

We can then calculate the brackets between the dynamical variables and find the only non-zero elements

$$\{x_i, p_j\} = \delta_{ij} \qquad \mathrm{i}\{\psi_{+i}, \psi_{+j}\} = \delta_{ij} \qquad \mathrm{i}\{\psi_{-i}, \psi_{-j}\} = \delta_{ij}.$$

From these we can finally determine the following algebra relations between all the conserved quantities:

$$\{H, H\} = \{H, Q\} = \{H, \tilde{Q}\} = \{H, R\} = \{H, \boldsymbol{L}\} = 0$$
$$\mathrm{i}\{Q, Q\} = \mathrm{i}\{\tilde{Q}, \tilde{Q}\} = 2H \quad \{Q, \tilde{Q}\} = 0 \quad \{Q, R\} = \tilde{Q} \quad \{\tilde{Q}, R\} = -Q$$
$$\{\boldsymbol{L}, Q\} = \{\boldsymbol{L}, \tilde{Q}\} = \{\boldsymbol{L}, R\} = 0 \quad \{L_i, L_j\} = \epsilon_{ijk} L_k \quad \{R, R\} = 0.$$

5. The Coulomb problem

The supersymmetric Coulomb problem, characterized by an $1/|\boldsymbol{x}|$ repulsive bosonic potential, can now be obtained from (1) by setting

$$\boldsymbol{U}(\boldsymbol{x}) = \frac{2^{\frac{1}{2}}}{|\boldsymbol{x}|^{\frac{1}{2}}}\hat{\boldsymbol{x}} \tag{18}$$

or equivalently by choosing $W(|\boldsymbol{x}|) = 2^{\frac{3}{2}}|\boldsymbol{x}|^{\frac{1}{2}}$ in (12)–(14). It follows that the equations of motion now read:

$$\ddot{\boldsymbol{x}} = \frac{\hat{\boldsymbol{x}}}{|\boldsymbol{x}|^2} - \mathrm{i}\frac{3}{2}\frac{2^{\frac{1}{2}}}{|\boldsymbol{x}|^{\frac{5}{2}}}\left(\boldsymbol{\psi}_+(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}}) + (\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})\boldsymbol{\psi}_- + (\boldsymbol{\psi}_+\cdot\boldsymbol{\psi}_-)\hat{\boldsymbol{x}} - \frac{7}{2}(\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}\right) \tag{19}$$

$$\dot{\boldsymbol{\psi}}_+ = -\frac{2^{\frac{1}{2}}}{|\boldsymbol{x}|^{\frac{3}{2}}}\left(\boldsymbol{\psi}_- - \frac{3}{2}(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}\right) \tag{20}$$

$$\dot{\boldsymbol{\psi}}_- = \frac{2^{\frac{1}{2}}}{|\boldsymbol{x}|^{\frac{3}{2}}}\left(\boldsymbol{\psi}_+ - \frac{3}{2}(\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}\right). \tag{21}$$

We will proceed to solve these equations for a Grassmann algebra with two generators in the next section.

The formulae for all the conserved quantities need not be written again for our special choice of $\boldsymbol{U}$ but it is worth mentioning the explicit form of the Hamiltonian

$$H = \frac{1}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} + \frac{1}{|\boldsymbol{x}|} - \mathrm{i}\frac{2^{\frac{1}{2}}}{|\boldsymbol{x}|^{\frac{3}{2}}}\left(\boldsymbol{\psi}_+\cdot\boldsymbol{\psi}_- - \frac{3}{2}(\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}})\right). \tag{22}$$

In the purely bosonic case the Coulomb potential admits a further well-known conserved quantity, the Runge–Lenz vector, given by

$$\boldsymbol{V}_{\mathrm{RL}} = [\boldsymbol{L}, \dot{\boldsymbol{x}}] - \hat{\boldsymbol{x}}.$$

Unfortunately, this extra conserved quantity does not seem to exist for the supersymmetric case. We will outline a proof for this in the appendix.

6. Explicit solutions

We will now solve equations (19)–(21) for the case of a Grassmann algebra with two generators. In other words, we assume that all physical quantities take their values in an algebra that is spanned by two Grassmann-odd elements ξ_1 and ξ_2, satisfying the relations

$$\xi_1^2 = \xi_2^2 = 0 \qquad \xi_1\xi_2 = -\xi_2\xi_1.$$

Using a method similar to that deployed in [9] we split the bosonic variable $\boldsymbol{x}$ into components according to the number of generators involved:

$$\boldsymbol{x}(t) = \boldsymbol{x}^0(t) + \boldsymbol{x}^{ij}(t)\,\mathrm{i}\xi_i\xi_j \equiv \boldsymbol{x}^0(t) + \boldsymbol{x}^1(t).$$

Taylor-expanding the potential function $\boldsymbol{U}$ then gives us

$$\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{U}(\boldsymbol{x}^0) + \nabla\boldsymbol{U}(\boldsymbol{x}^0)\boldsymbol{x}^1.$$

Note that all quantities are vector-valued. We will indicate components, if necessary, by lower indices.

The fermionic variables are time-dependent multiples of ξ_1 and ξ_2 since for the particular Grassmann algebra chosen there can be no product of three or more generators. Therefore, in products of bosonic and fermionic quantities only bosonic terms of zeroth order in the generators contribute. Writing

$$\boldsymbol{\psi}_+(t) = \boldsymbol{\psi}_+^{\mathbf{1}}(t)\xi_1 + \boldsymbol{\psi}_+^{\mathbf{2}}(t)\xi_2 \qquad \boldsymbol{\psi}_-(t) = \boldsymbol{\psi}_-^{\mathbf{1}}(t)\xi_1 + \boldsymbol{\psi}_-^{\mathbf{2}}(t)\xi_2$$

both fermionic variables decompose into two terms with *real-valued* time-dependent coefficients ψ_+^I and ψ_-^I, respectively, where $I = 1, 2$.

The equations of motion for the coefficients are similar to (20) and (21)—the only differences being that ψ_+ and ψ_- have to be replaced by the component functions ψ_+^I and ψ_-^I and x has to be replaced by x^0.

While the fermionic equations thus do not look much different after the decomposition the bosonic equation (19) splits into two, namely,

$$\ddot{x}^0 = \frac{\hat{x}^0}{|x^0|^2} \tag{23}$$

$$\ddot{x}^1 = \frac{x^1 - 3(x^1 \cdot \hat{x}^0)\hat{x}^0}{|x^0|^3} + a(x^0, \psi_+, \psi_-) \tag{24}$$

where

$$a(x^0, \psi_+, \psi_-) = -\mathrm{i}\frac{3}{2}\frac{2^{\frac{1}{2}}}{|x^0|^{\frac{5}{2}}}\bigg(\psi_+(\psi_- \cdot \hat{x}^0) + (\psi_+ \cdot \hat{x}^0)\psi_- + (\psi_+ \cdot \psi_-)\hat{x}^0 - \frac{7}{2}(\psi_+ \cdot \hat{x}^0)(\psi_- \cdot \hat{x}^0)\hat{x}^0\bigg)$$

does not contain x^1.

Following [9] we will now systematically solve these equations by working 'upwards' from the bottom layer of the system (23) through the fermion equations (20) and (21) to the top layer (24).

6.1. The bottom layer bosonic equation

The lowest order equation (23) is of course just the familiar equation of the Coulomb problem. Its solutions are the well-known hyperbolae in the plane orthogonal to the angular momentum vector L (if we set apart for a moment the special case $L = 0$). In order to make use of the solutions for the remaining equations it will be necessary to write them in an explicit time-dependent fashion. To do this we utilize a method first devised by Moser in [11] for the Kepler problem with a negative total energy and then extended to the positive energy case by Belbruno [12].

We can summarize these results adapted to our problem by saying that there is a diffeomorphism between the constant energy surface $H = E^0$ in the phase space of the Coulomb problem and the tangent bundle of the upper (or lower) sheet of the hyperboloid $\mathcal{H}$ specified in four-dimensional Euclidean space by $X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1$. This diffeomorphism takes geodesics on $\mathcal{H}$ into solutions to (23) requiring only a change of the time variable.

More explicitly, if we parametrize a geodesic on $\mathcal{H}$ by $X_i(s)$, $i = 0, \ldots, 3$, and denote by $P_i(s)$, $i = 0, \ldots, 3$, its tangent then

$$x^0(s) = P(s)(1 + X_0(s)) - X(s)P_0(s) \tag{25}$$

$$p^0(s) = X(s)\frac{1}{1 + X_0(s)} \tag{26}$$

transforms this geodesic into a solution of the Kepler problem for the energy $E^0 = \frac{1}{2}$, when we assume the following transcendental relationship between the geodesic 'time'-parameter s and physical time t:

$$t = \int_0^s |x^0(s')|\,\mathrm{d}s'. \tag{27}$$

Solutions for arbitrary energy can then be obtained by the scaling

$$\tilde{\boldsymbol{x}}^0 = \frac{1}{2E^0}\boldsymbol{x}^0 \qquad \tilde{\boldsymbol{p}}^0 = (2E^0)^{\frac{1}{2}}\boldsymbol{p}^0 \qquad \tilde{t} = \frac{1}{(2E^0)^{\frac{3}{2}}}t. \tag{28}$$

Every geodesic on $\mathcal{H}$ can be mapped using an appropriate $SO(3,1)$-transformation into

$$\boldsymbol{X}(s) = (\cosh s \cosh\beta, \sinh s, 0, -\cosh s \sinh\beta).$$

For the Coulomb problem this equates to choosing the plane of motion and the orientation of the hyperbola in it. (Note that for the specific parametrization chosen we have also fixed the origin of time.) Specifically, we get

$$\boldsymbol{x}^0 = \left(e + \cosh s, -(e^2-1)^{\frac{1}{2}}\sinh s, 0\right) \tag{29}$$

$$\boldsymbol{p}^0 = \left(\frac{\sinh s}{1+e\cosh s}, -\frac{(e^2-1)^{\frac{1}{2}}\cosh s}{1+e\cosh s}, 0\right) \tag{30}$$

where $e = \cosh\beta$ represents the eccentricity of the hyperbola. The relationship between s and t is given by

$$t = e\sinh s + s. \tag{31}$$

This equation cannot be inverted analytically.

One can read off from (29) that the motion is indeed hyperbolic, taking place in the x_1–x_2-plane with the focal points on the x_1-axis and the origin of time chosen such that $t = 0$ at the point of closest approach to the potential centre. We just mention that $\boldsymbol{L}$ is oriented in $(-x_3)$-direction and (for constant energy) $|\boldsymbol{L}| = (e^2-1)^{\frac{1}{2}}$; the scattering angle θ is determined by $e\sin\frac{\theta}{2} = 1$.

There are two special cases worth mentioning, namely $e = 1$ when the scattering angle θ is π and the angular momentum is zero and $e \longrightarrow \infty$ when $\theta = 0$ and the equations describe free motion at an infinite distance from the centre of the potential.

6.2. *The fermionic equations*

To solve equations (20) and (21) for $\boldsymbol{x} \equiv \boldsymbol{x}^0$ we can first go back to our general solutions (10) and (11). They state in this circumstance that ψ_+ and ψ_- are linear combinations of $\dot{\boldsymbol{x}}^0$ and $\boldsymbol{U}(\boldsymbol{x}^0)$ with Grassmann-odd coefficients. (Note that we can safely assume $E^0 > 0$ due to the repulsive nature of the Coulomb potential.)

To analyse the fermionic movement it is therefore sensible to look at the motion of $\dot{\boldsymbol{x}}^0$ and $\boldsymbol{U}(\boldsymbol{x}^0)$. The first interesting aspect is that if we combine velocity and potential in a six-dimensional vector $Z = (\dot{\boldsymbol{x}}^0, \boldsymbol{U})$ then the motion of this vector takes place on S^3, since

$$\dot{\boldsymbol{x}}^0 \cdot \dot{\boldsymbol{x}}^0 + \boldsymbol{U} \cdot \boldsymbol{U} = 2E^0 = 1$$

for our particular solution, but $\dot{x}_3^0 = U_3 = 0$ trivially. This is in fact very similar to the one-dimensional problem where motion takes place on S^1 instead [9].

If we project the motion into the $\dot{x}_1^0$–$\dot{x}_2^0$-plane then this two-dimensional vector describes a circle with centre at $\left(0, -e/(e^2-1)^{\frac{1}{2}}\right)$ (see figure 1). The motion starts for $t = -\infty$ at the intersection of this circle with S^1, running initially towards the origin but then bending towards the $\dot{x}_2^0$-axis and reaching its nearest point to the origin at $(0, -(e^2-1)^{\frac{1}{2}}/(e+1))$. Then it turns back outwards and runs symmetrically to the other intersection point, reaching it at $t = \infty$.

The projection into the U_1–U_2-plane shows a different picture. Due to the factor $1/|\boldsymbol{x}^0|^{\frac{1}{2}}$ in (18) the motion starts and ends at the origin. It runs radially outwards along an asymptote

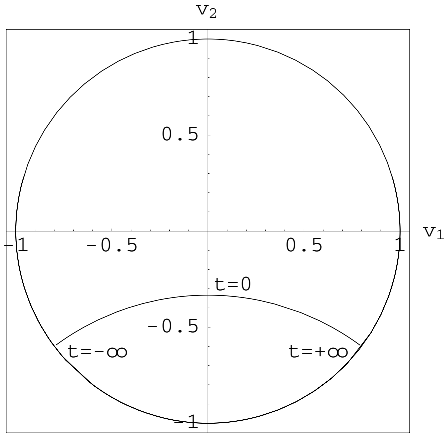

Figure 1. Projection of the bosonic motion into the $\dot{x}_1^0$–$\dot{x}_2^0$-plane $\left(e = \frac{5}{4}\right)$.

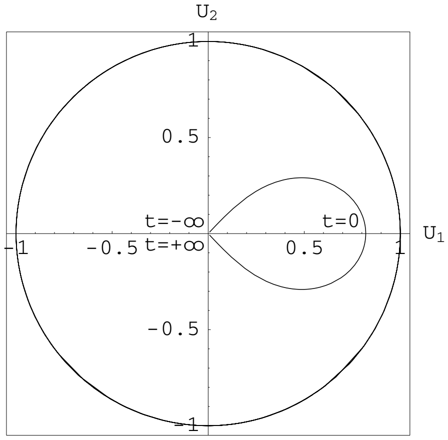

Figure 2. Projection of the bosonic motion into the U_1–U_2-plane $\left(e = \frac{5}{4}\right)$.

determined by the angular momentum but then bends towards the U_1-axis, crossing it at time $t = 0$ and returns to the origin in a symmetric way (see figure 2).
When we come back to solutions (10) and (11) we can immediately state that

$$\lim_{t\to\infty} \psi_+ = \lim_{t\to-\infty} \psi_+ = Q\dot{x}^0 \qquad \lim_{t\to\infty} \psi_- = \lim_{t\to-\infty} \psi_- = \tilde{Q}\dot{x}^0$$

so that we know the asymptotic behaviour of these solutions.

However, we can visualize the fermionic motion completely if we refer to the bosonic motion on S^3. For this it is more convenient to use the real-valued components ψ_+^I and ψ_-^I than the Grassmann-valued quantities. One advantage is that we can meaningfully define the length of these vectors: for example,

$$S^I = \tfrac{1}{2}\left(\psi_+^I\right)^2 + \tfrac{1}{2}\left(\psi_-^I\right)^2$$

is a conserved quantity arising from the invariance of the component equations of motion under change of generators. Note that because we are dealing with real-valued quantities now, none of the two products is zero.

To understand the close connection between the above-mentioned bosonic motion on S^3 and fermionic motion let us ignore the third components $\psi_{+3}^{\ I}$ and $\psi_{-3}^{\ I}$ for a moment, since, as we will see later, their equations decouple. Combining the remaining fermionic terms into the vector-valued object

$$\Psi^I = \left(\psi_{+1}^{\ I}, \psi_{+2}^{\ I}, \psi_{-1}^{\ I}, \psi_{-2}^{\ I}\right)$$

we see that Ψ^I lies in the plane spanned by the bosonic vector $Z = (\dot{\boldsymbol{x}}^0, \boldsymbol{U})$ and one specific orthogonal vector $Z_\perp = (-\boldsymbol{U}, \dot{\boldsymbol{x}}^0)$ obtained by rotating Z by 90° in the $\dot{x}_i$–U_i-planes for $i = 1, 2$. (We are neglecting the trivial third components here.) The angle between Ψ^I and the two bosonic vectors is entirely specified by the two supercharges Q and $\tilde{Q}$. (To be more precise it is specified by their real-valued components Q^I and $\tilde{Q}^I$ with respect to the two Grassmann generators. For simplicity, we will not indicate this difference in our notation assuming that the precise meaning is always clear from the context.) In other words, as the bosonic vector Z moves on S^3, determined by the bosonic equation of motion, the fermionic vector simply follows this movement rigidly, its direction being fixed with respect to the bosonic vector and its orthogonal complement by the two supercharges.

This result is very similar to the one-dimensional case where one finds similarly constructed bosonic and fermionic vectors corotating on S^1. Still, there is a major difference in more than one dimension. This lies in the fact that Q and $\tilde{Q}$ do not specify the fermionic solutions uniquely. The solution in one dimension can be derived by inverting equations (7) and (8), yet in two or more dimensions this inversion is no longer possible. Instead, these two equations only specify the two angles between Ψ^I and the *two* bosonic vectors $Z, Z_\perp$ but not more. This means that the general Ψ^I (again neglecting third components) moves simultaneously on two regular four-dimensional cones with centre at the origin and their symmetry axes Z and $Z_\perp$, respectively. As Z and $Z_\perp$ move with time Ψ^I follows rigidly attached to the surface of both cones and thus lies on the intersection of them. In the particular solution that we gave above this intersection corresponds just to one point (taken at fixed distance from the origin), thus specifying this particular solution completely. However, in general the intersection of the two cones (and S^3) will yield a one-dimensional sphere (just as the intersection of two three-dimensional cones and S^2 is equivalent to a zero-dimensional sphere, i.e. two discrete points). All points on this sphere have the same combination of supercharges Q and $\tilde{Q}$, which means that there exists a one-dimensional space of degrees of freedom for the fermionic motion that is not determined by the supercharges.

In fact, we will now derive a second solution to (20) and (21) that fixes the motion of Ψ^I on this one-dimensional space. Since the particular solution involving the supercharges described above lies entirely in the Z–$Z_\perp$-plane we will now look for a solution in the plane orthogonal to it. (Remember that the space we are considering is four-dimensional.) This plane can be spanned by the two vectors

$$Y = \left(\dot{x}_2^0, -\dot{x}_1^0, -U_2, U_1\right) \qquad Y_\perp = \left(U_2, -U_1, \dot{x}_2^0, -\dot{x}_1^0\right).$$

A straightforward ansatz for the second solution is $\Psi = PY + \tilde{P}Y_\perp$, where P and $\tilde{P}$ are Grassmann-odd constants. Yet, this ansatz turns out to be too simple and does not yield a solution. Instead, we have to allow for time-dependent coefficients of Y and $Y_\perp$. In other words, we make the assumption that

$$\Psi = \lambda_1(t)Y + \lambda_2(t)Y_\perp \tag{32}$$

where the λ_i are Grassmann-odd functions of time which still have to be determined. Inserting this ansatz into (20) and (21) we find that the equations of motion are satisfied as long as the coefficient functions $\lambda_i(t)$ obey the condition

$$\dot{\lambda}_i = -\frac{1}{2^{\frac{1}{2}}|\boldsymbol{x}^0|^{\frac{3}{2}}}\epsilon_{ij}\lambda_j. \tag{33}$$

This first-order differential equation for λ_i can be solved by

$$\lambda_1(t) = P\cos\omega(t) - \tilde{P}\sin\omega(t) \tag{34}$$

$$\lambda_2(t) = P\sin\omega(t) + \tilde{P}\cos\omega(t) \tag{35}$$

where P and $\tilde{P}$ are two Grassmann-odd constants and

$$\omega(t) = \int_0^t \frac{1}{2^{\frac{1}{2}}|\boldsymbol{x}^0|^{\frac{3}{2}}}\,\mathrm{d}t' = \frac{1}{2^{\frac{1}{2}}}\int_0^{s(t)} \frac{1}{(1+e\cosh s)^{\frac{1}{2}}}\,\mathrm{d}s. \tag{36}$$

This integral cannot be evaluated analytically but it can be written in terms of the standard first elliptic integral $F(s,k)$ as

$$\omega(t) = \frac{-2^{\frac{1}{2}}\mathrm{i}}{(e+1)^{\frac{1}{2}}}F\left(\frac{\mathrm{i}}{2}s(t), \left(\frac{2e}{e+1}\right)^{\frac{1}{2}}\right).$$

We can immediately read off from (36) that $\omega(t)$ is a monotonically growing function that converges to a constant as $t \longrightarrow \pm\infty$ and that has only one zero at $t = 0$. The asymptotic behaviour of ω can be described by the formula

$$\omega_\infty = \lim_{t\to+\infty}\omega(t) = -\lim_{t\to-\infty}\omega(t) = \frac{\pi}{2e^{\frac{1}{2}}}P_{-\frac{1}{2}}\left(\frac{1}{e}\right)$$

where $P_{\frac{1}{2}}(k)$ is the Legendre function of first kind.

There are two interesting limiting cases, namely $e = 1$ and $e \to \infty$. Because $P_{-1/2}(1) = 1$ and $P_{-1/2}(0)$ is a finite constant we find that $\omega \to \frac{\pi}{2}$ in the first and $\omega \to 0$ in the second case. In other words, ω_∞ is a monotonically falling function of e, taking values in the interval $\left(0, \frac{\pi}{2}\right]$.

Going back to equations (34) and (35) and inserting them into our ansatz (32) we thus find the result

$$\Psi = P(\cos\omega(t)Y + \sin\omega(t)Y_\perp) + \tilde{P}(-\sin\omega(t)Y + \cos\omega(t)Y_\perp) \tag{37}$$

which can be interpreted as follows (substituting the real components Ψ^I, P^I, $\tilde{P}^I$ for the Grassmann-valued quantities, if necessary): in addition to motion in the Z–$Z_\perp$-plane the fermionic vector can also move in the plane orthogonal to it. As in the previous case it is rigidly attached to and corotating with two orthogonal bosonic vectors. The motion of these two vectors, however, is slightly more complicated than before. They are not rigidly connected to Z and $Z_\perp$ in a vierbein but rather have a time-dependent phase (unless the eccentricity tends to infinity so that our formulae describe a freely moving particle at infinite distance from the origin, in which case $\omega(t) \equiv 0$.)

This phase changes in a continuous fashion between $-\omega_\infty$ and ω_∞, where ω_∞ depends on the eccentricity of the bosonic solution and takes its maximum value $\frac{\pi}{2}$ in the case $e = 1$, i.e. in the absence of the angular momentum.

Thus the second solution describes a circular motion orthogonal to the first but with a time-dependent phase difference with respect to the rigid frame provided by the bosonic vector Z and its three chosen orthogonals. We can think of this motion as parametrizing the motion on S_1 that we mentioned above.

So far we have neglected the third components of ψ_+ and ψ_-, those in the chosen direction of angular momentum, for which the bosonic movement is trivial (i.e. $\dot{x}_3 = U_3 = 0$). It was one of the results from [9] that triviality of the bosonic solution does not extend to the fermionic solution and the same can be observed in the multidimensional case.

Since the bosonic motion takes place in the x_1–x_2-plane and therefore $x_3 \equiv 0$ equations (20) and (21) simplify considerably:

$$\dot{\psi}_{+3} = -\frac{2^{\frac{1}{2}}}{|\boldsymbol{x}^0|^{\frac{3}{2}}}\psi_{-3} \qquad \dot{\psi}_{-3} = \frac{2^{\frac{1}{2}}}{|\boldsymbol{x}^0|^{\frac{3}{2}}}\psi_{+3}. \tag{38}$$

Incidentally, up to a factor of two these are the same equations as (33) and therefore have (again up to a factor of two) the same solutions:

$$\psi_{+3}(t) = O\cos 2\omega(t) - \tilde{O}\sin 2\omega(t) \tag{39}$$

$$\psi_{-3}(t) = O\sin 2\omega(t) + \tilde{O}\cos 2\omega(t) \tag{40}$$

where O and $\tilde{O}$ are two further Grassmann-odd constants and $\omega(t)$ is the same function as before. The only difference consists of the extra factor of two and is responsible for a slight change in the asymptotic behaviour: whereas the maximum phase obtained for $t \to \pm\infty$ still converges to zero for $e \to \infty$ implying that $\psi_{+3} = O$ and $\psi_{-3} = \tilde{O}$ are both constants, this maximum phase now tends to π rather than $\frac{\pi}{2}$ for $e = 1$ meaning that for zero angular momentum ψ_{+3} and ψ_{-3} describe a full circle.

So although there is no bosonic motion in the third direction (to lowest order) the fermions somehow 'see' the bosonic potential and behave accordingly. In fact, we can think of (38) as fermionic equation of motion for a one-dimensional Coulomb potential—the only difference is that the strength of this potential is quadrupled with respect to the original problem. To see this we have to interpret the right-hand sides of (38) as $\mp\tilde{U}'(x)\psi_{\mp}$ where $\tilde{U}'(x)$ is given by

$$\tilde{U}'(x) = \frac{2^{\frac{1}{2}}}{x^{\frac{3}{2}}} \quad\Longrightarrow\quad \tilde{U}(x) = -\frac{2^{\frac{3}{2}}}{x^{\frac{1}{2}}}.$$

Thus the imagined one-dimensional bosonic potential has to be

$$\tilde{V}(x) = \frac{1}{2}\tilde{U}^2 = \frac{4}{x} = 4V.$$

This can be seen as an explanation for the extra factor of two appearing in the solutions (39) and (40).

We want to conclude this section by a short discussion of the newly found fermionic constants P, $\tilde{P}$, O and $\tilde{O}$ since there is a major difference between them and the supercharges Q and $\tilde{Q}$. Formally we can write all four constants as conserved quantities by simply inverting equations (32), (34), (35), (39) and (40). However, they cannot be seen as an original symmetry of the Lagrangian, in the sense that they cannot be derived as Noether charges by a variation just involving the dynamical variables. On the contrary, to derive those charges from the Lagrangian we have to presuppose a certain knowledge of the system, e.g. the orientation of angular momentum, to find the correct variation. As an example, O and $\tilde{O}$ can be formally found as conserved quantities using the following variation:

$$\begin{aligned} \delta\psi_{+3} &= -\epsilon\cos 2\omega(t) & \delta\psi_{-3} &= -\epsilon\sin 2\omega(t) \\ \tilde{\delta}\psi_{+3} &= \epsilon\sin 2\omega(t) & \tilde{\delta}\psi_{-3} &= -\epsilon\cos 2\omega(t). \end{aligned}$$

Evidently, neither function on the right-hand side is a function of the dynamical variables only, instead we find a rather complicated function of time involving $\omega(t)$. In addition, this variation would have to be different if we would alter the initial data of the bosonic motion, for example, by making a different choice for the orientiation of angular momentum. Thus P, $\tilde{P}$, O and $\tilde{O}$ should be seen as 'lesser' charges, not comparable with the two supercharges Q and $\tilde{Q}$ which are genuine Noether charges as we have seen above.

There are, however, some interesting observations about the interrelation of these extra fermionic quantities to be made. Firstly, one can read off from (39) and (40) that the product

$S = \mathrm{i}\psi_{+3}\psi_{-3}$ or, more generally, $S = \mathrm{i}(\boldsymbol{L}^0 \cdot \boldsymbol{\psi}_+)(\boldsymbol{L}^0 \cdot \boldsymbol{\psi}_-)$ is conserved, where $\boldsymbol{L}^0$ denotes the bottom layer angular momentum. This is a consequence of the symmetry

$$\delta\boldsymbol{\psi}_+ = \epsilon(\boldsymbol{L}^0 \cdot \boldsymbol{\psi}_-)\boldsymbol{L}^0 \qquad \delta\boldsymbol{\psi}_- = \epsilon(\boldsymbol{L}^0 \cdot \boldsymbol{\psi}_+)\boldsymbol{L}^0.$$

Since we have restricted ourselves to two Grassmann generators we can replace $\boldsymbol{L}^0$ simply by $\boldsymbol{L}$ and so find a genuine symmetry of the Lagrangian, involving only the dynamical quantities. Moreover, between all six fermionic constants mentioned there then exists the remarkable relation

$$\mathrm{i}(Q\tilde{Q} + P\tilde{P} + O\tilde{O}) = 2ER$$

where, as we have seen, the bosonic constants R and E are genuine Noether charges for arbitrary Grassmann algebra.

6.3. *The top layer bosonic equation*

We finally have to solve equation (24) for the bosonic top layer, concentrating first on the homogeneous part of this equation, i.e. ignoring $\boldsymbol{a}(\boldsymbol{x}^0, \boldsymbol{\psi}_+, \boldsymbol{\psi}_-)$.

In the one-dimensional case there were two solutions to the analogue of this equation, namely,

$$x^1 = c\dot{x}^0 \qquad x^1 = c\dot{x}^0 \int_{t_0}^{t} \frac{1}{(\dot{x}^0)^2}\,\mathrm{d}t'.$$

Whereas the first term is still a solution to the three-dimensional problem, the second one is not. A more fruitful generalization from the one- to the multidimensional case is given by the idea that solutions to the top layer equation describe variations of the bottom layer system with the free constants of that motion. We will explain this idea in more detail below.

To find an explicit form for the solutions we try the following ansatz:

$$\boldsymbol{x}^1(t) = f_1(t)\boldsymbol{x}^0(t) + f_2(t)\dot{\boldsymbol{x}}^0(t). \tag{41}$$

Inserting this ansatz into (24) and using the bottom layer equation (23) we can evaluate both sides and write them as time-dependent linear combinations of $\boldsymbol{x}^0$ and $\dot{\boldsymbol{x}}^0$. Comparison of the coefficients then yields the two relations

$$\ddot{f}_1(t) + \frac{2}{|\boldsymbol{x}^0|^3}\dot{f}_2(t) = -3\frac{f_1(t)}{|\boldsymbol{x}^0|^3} \tag{42}$$

$$2\dot{f}_1(t) + \ddot{f}_2(t) = 0. \tag{43}$$

Integrating equation (43) and inserting the result

$$\dot{f}_2(t) = -2f_1(t) + c \tag{44}$$

into (42) we get the following equation for $f_1(t)$:

$$\ddot{f}_1(t) = \frac{1}{|\boldsymbol{x}^0|^3}(f_1(t) - 2c). \tag{45}$$

We can recognize that the homogeneous part of this equation is nothing else than the bottom layer equation and this means that we can write its two solutions as the two component solutions of (23) given in (29). Since the inhomogeneous solution is just the constant $2c$ we can write

$$f_1(t) = C_1x_1^0(t) + C_2x_2^0(t) + 2c = C_1(e + \cosh s(t)) - C_2(e^2 - 1)^{\frac{1}{2}}\sinh s(t) + 2c.$$

Applying this result to (44) we find that

$$f_2(t) = -2C_1X_1(t) - 2C_2X_2(t) - 3ct + \tilde{c}$$

where

$$X_1(t) = \int_0^t x_1^0(t')\,\mathrm{d}t' = \frac{3}{2}es(t) + \frac{1}{2}e\sinh s(t)\cosh s(t) + (e^2-1)\sinh s(t)$$

$$X_2(t) = \int_0^t x_2^0(t')\,\mathrm{d}t' = -(e^2-1)^{\frac{1}{2}}\left(\cosh s(t) + \frac{1}{2}e\sinh^2 s(t) - 1\right).$$

Inserting these results back into our ansatz (41) we find that

$$\begin{aligned}\boldsymbol{x}^1(t) = {} & C_1\left(x_1^0(t)\boldsymbol{x}^0(t) - 2X_1(t)\dot{\boldsymbol{x}}^0(t)\right) + C_2\left(x_2^0(t)\boldsymbol{x}^0(t) - 2X_2(t)\dot{\boldsymbol{x}}^0(t)\right)\\ & + c(2\boldsymbol{x}^0(t) - 3t\dot{\boldsymbol{x}}^0(t)) + \tilde{c}\dot{\boldsymbol{x}}^0(t).\end{aligned} \tag{46}$$

There are still two solutions to (24) missing. As all solutions found so far lie in the plane of motion spanned by $\boldsymbol{x}^0(t)$ and $\dot{\boldsymbol{x}}^0(t)$ we have to look now at solutions in the third direction, i.e. for $x_3^1(t)$. The equation of motion simplifies here to

$$\ddot{x}_3^1(t) = \frac{1}{|\boldsymbol{x}^0|^3}x_3^1(t)$$

since $x_3^0(t) \equiv 0$. Again this is just the bottom layer equation of motion, solved by $x_1^0(t)$ and $x_2^0(t)$. The general solution to the homogeneous part of (24) is thus given by (46) and

$$\boldsymbol{x}^1(t) = \tilde{C}_1 x_1^0(t)\boldsymbol{L}^0 + \tilde{C}_2 x_2^0(t)\boldsymbol{L}^0. \tag{47}$$

When we look for an interpretation of these solutions the analogy with the one-dimensional case will be, as mentioned, quite fruitful. In one dimension the solution to the homogeneous part of the x^1-equation could be written as a sum of variations:

$$x^1(t) = c_1\frac{\delta x^0}{\delta t_0}(t) + c_2\frac{\delta x^0}{\delta E}(t).$$

The same turns out to be true in the three-dimensional case. To start with, the term with coefficient $\tilde{c}$ in (46) can be clearly interpreted as variation of our bottom layer solution with initial time t_0:

$$\dot{\boldsymbol{x}}^0(t) = \frac{\delta\boldsymbol{x}^0}{\delta t_0}.$$

Variation with energy can also be found in the solution for $\boldsymbol{x}^1$ as can be seen in the following way: while solution (29) is valid only for energy $E = \frac{1}{2}$ we can obtain a solution for arbitrary energy E by scaling both $\boldsymbol{x}^0$ and t according to (28). The variation of $\boldsymbol{x}^0$ with energy is then obtained by comparing $\boldsymbol{x}_E^0(t)$ with $\boldsymbol{x}_{E+\mathrm{d}E}^0(t)$:

$$\frac{\delta\boldsymbol{x}_E^0}{\delta E} = \frac{\partial\boldsymbol{x}_E^0}{\partial E} + \frac{\partial\boldsymbol{x}_E^0}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}E} = -\frac{1}{2E^2}\boldsymbol{x}^0 + 3(2E)^{\frac{1}{2}}t_E\dot{\boldsymbol{x}}_E^0.$$

Evaluating this result at $E = \frac{1}{2}$ yields

$$\left.\frac{\delta\boldsymbol{x}_E^0}{\delta E}\right|_{E=\frac{1}{2}} = -2\boldsymbol{x}^0 + 3t\dot{\boldsymbol{x}}^0$$

leaving us with the term in (46) with the coefficient c. It is quite important to realize that we have to compare two solutions of different energies at the *same* physical time t, i.e. we must not change clocks and use t_E as a parameter for the solution $\boldsymbol{x}_{E+\mathrm{d}E}^0$.

These two variations comprised the homogeneous part of the solution to the one-dimensional problem. For the Coulomb problem there are four more independent ways of varying the bottom layer solution. One of these consists of rotating the hyperbolic orbit in the plane of motion by a small angle ϕ around the angular momentum axis. The variation is thus given by

$$\frac{\delta \boldsymbol{x}^0}{\delta \phi} = \left(-x_2^0, x_1^0, 0\right).$$

This variation coincides with a linear combination of two terms in (46), namely,

$$\frac{\delta \boldsymbol{x}^0}{\delta \phi} = -\frac{e}{e^2-1}\left[\left(x_2^0 \boldsymbol{x}^0 - 2X_2\dot{\boldsymbol{x}}^0\right) + (e^2-1)^{\frac{1}{2}}\left(e^{\frac{1}{2}} + e^{-\frac{1}{2}}\right)^2 \dot{\boldsymbol{x}}^0\right].$$

We can thus interpret the term with coefficient C_2 in (46) as a combination of rotation in the plane of motion and variation of the initial time parameter.

Another way of varying $\boldsymbol{x}^0$ is to change the eccentricity of the hyperbola which is explicitly included as a parameter in the solution (29). We find

$$\frac{\delta \boldsymbol{x}^0}{\delta e} = \frac{\partial \boldsymbol{x}^0}{\partial e} + \frac{\partial \boldsymbol{x}^0}{\partial s}\frac{\partial s}{\partial e} = \left(1 - \frac{x_2^0 \dot{x}_1^0}{L_3^0}, \frac{x_1^0 \dot{x}_1^0}{L_3^0}, 0\right)$$

where we have used that

$$\frac{\partial s}{\partial e} = \left(\frac{\partial e}{\partial s}\right)^{-1}\Bigg|_{t=\mathrm{const}} = -\frac{\sinh s}{1 + e\cosh s}.$$

We again find this variation as a linear combination of two terms in (46), namely,

$$\frac{\delta \boldsymbol{x}^0}{\delta e} = -\frac{1}{e^2-1}\left[\left(x_1^0 \boldsymbol{x}^0 - 2X_1\dot{\boldsymbol{x}}^0\right) - e(2\boldsymbol{x}^0 - 3t\dot{\boldsymbol{x}}^0)\right]$$

and so the C_1-term in (46) describes a combination of variation with eccentricity and energy. It is now to be expected that the two remaining terms of the homogeneous $\boldsymbol{x}^1$-solution are connected to a tilt of the plane of motion or, equivalently, to a change in the orientation of angular momentum $\boldsymbol{L}^0$:

$$\delta \boldsymbol{L}^0 = \delta L_1^0 \boldsymbol{e}_1 + \delta L_2^0 \boldsymbol{e}_2$$

here $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ represent unit vectors in the respective directions. We find that

$$\frac{\delta \boldsymbol{x}^0}{\delta L_1^0} = x_1^0 \boldsymbol{e}_3 \qquad \frac{\delta \boldsymbol{x}^0}{\delta L_2^0} = x_2^0 \boldsymbol{e}_3$$

which are evidently proportional to the two terms in (47) since $\boldsymbol{L}^0$ is oriented in the third direction in our particular solution to the bottom layer system.

Thus we have now derived and found an interpretation for all the terms of the homogeneous solution to the top layer system. As was the case for one-dimensional systems this solution describes all the possible variations of the bottom layer with all possible parameters describing this layer. In our case we need six parameters to describe the motion of the original Coulomb problem (initial time, energy, eccentricity, orientation of major axis and two parameters describing the plane of motion) and correspondingly there are six independent solutions to the top layer.

The last step is now to find one particular solution to the inhomogeneous equation (24). The general idea is to take linear combinations of the inhomogeneous terms a_i, integrate them over time and multiply them by one of the solutions $\boldsymbol{x}^1_{\mathrm{hom}}$ of the homogeneous equation, then sum over all solutions:

$$\boldsymbol{x}^1_{\mathrm{inhom}} = \sum_{\alpha=1}^{n} \boldsymbol{x}^1_{\mathrm{hom},\alpha} \int_0^t f_{\alpha,i}(t')a_i(t')\,\mathrm{d}t'.$$

We determine the coefficient functions $f_{\alpha,i}$ by two conditions: firstly, in taking the time derivative we want to get

$$\dot{\boldsymbol{x}}^1_{\text{inhom}} = \sum_{\alpha=1}^{n} \dot{\boldsymbol{x}}^1_{\text{hom},\alpha} \int_0^t f_{\alpha,i}(t')a_i(t')\,\mathrm{d}t'$$

i.e. all the terms involving derivatives of the integrals must cancel each other. Then for the second time derivative we arrive at

$$\ddot{\boldsymbol{x}}^1_{\text{inhom}} = \sum_{\alpha=1}^{n} \ddot{\boldsymbol{x}}^1_{\text{hom},\alpha} \int_0^t f_{\alpha,i}(t')a_i(t')\,\mathrm{d}t' + \left(\sum_{\alpha=1}^{n} \dot{\boldsymbol{x}}^1_{\text{hom},\alpha} f_{\alpha,i}(t)\right) a_i(t).$$

Because the $\boldsymbol{x}^1_{\text{hom},\alpha}$-terms satisfy the homogeneous equation the first sum will give the homogeneous part of (24). For the next term to give the inhomogeneous part we need the second condition that the sum in parenthesis is equal to the unit vector $\boldsymbol{e}^i$.

We will now demonstrate this procedure: for simplicity, we begin with the third component of $\boldsymbol{x}^1$. Since it decouples from the other components we can safely assume $f_{\alpha,1} = f_{\alpha,2} = 0$, i.e. writing f_α instead of $f_{\alpha,3}$ we can make the ansatz

$$x^1_{\text{inhom},3} = x^0_1 \int_0^t f_1(t')a_3(t')\,\mathrm{d}t' + x^0_2 \int_0^t f_2(t')a_3(t')\,\mathrm{d}t'. \tag{48}$$

Our first condition then yields $x^0_1(t)f_1(t)a_3(t) + x^0_2(t)f_2(t)a_3(t) = 0$ and thus we can set $f_1 = cx^0_2$ and $f_2 = -cx^0_1$. Then we find

$$\begin{aligned}\ddot{x}^1_{\text{inhom},3} &= c\left(\ddot{x}^0_1 \int_0^t x^0_2(t')a_3(t')\,\mathrm{d}t' - \ddot{x}^0_2 \int_0^t x^0_1(t')a_3(t')\,\mathrm{d}t'\right) + c\left(\dot{x}^0_1 x^0_2 - \dot{x}^0_2 x^0_1\right)a_3(t)\\ &= \frac{1}{|\boldsymbol{x}^0|^3} c\left(x^0_1 \int_0^t x^0_2(t')a_3(t')\,\mathrm{d}t' - x^0_2 \int_0^t x^0_1(t')a_3(t')\,\mathrm{d}t'\right) - cL^0_3 a_3(t)\end{aligned}$$

and so by equating $c = -\frac{1}{L^0_3}$ we end up with the required result, namely,

$$\ddot{x}^1_{\text{inhom},3} = \frac{1}{|\boldsymbol{x}^0|^3} x^1_3 + a_3(t).$$

Since the equations for the first and second components are coupled we need all four remaining homogeneous solutions to give the correct inhomogeneous term. We just write the result

$$\begin{aligned}\boldsymbol{x}^1{}_{\text{inhom}} = \frac{1}{L^0_3}\Bigg[&\dot{\boldsymbol{x}}^0 \int_0^t a_1\left(2\frac{X_1}{L^0_3}\dot{x}^0_2 x^0_2 + 2\frac{X_2}{L^0_3}\left(L^0_3 - \dot{x}^0_1 x^0_2\right) - 3t x^0_2\right)\\ &+ a_2\left(2\frac{X_2}{L^0_3}\dot{x}^0_1 x^0_1 - 2\frac{X_1}{L^0_3}\left(L^0_3 + \dot{x}^0_2 x^0_1\right) + 3t x^0_1\right)\mathrm{d}t'\\ &+ (2\boldsymbol{x}^0 - 3t\dot{\boldsymbol{x}}^0)\int_0^t \left(-a_1 x^0_2 + a_2 x^0_1\right)\mathrm{d}t'\\ &+ \left(x^0_1\boldsymbol{x}^0 - 2X_1\dot{\boldsymbol{x}}^0\right)\int_0^t \left(a_1\frac{\dot{x}^0_2 x^0_2}{L^0_3} - a_2\left(1 + \frac{\dot{x}^0_2 x^0_1}{L^0_3}\right)\right)\mathrm{d}t'\\ &+ \left(x^0_2\boldsymbol{x}^0 - 2X_2\dot{\boldsymbol{x}}^0\right)\int_0^t \left(a_1\left(1 - \frac{\dot{x}^0_1 x^0_2}{L^0_3}\right) + a_2\frac{\dot{x}^0_1 x^0_1}{L^0_3}\right)\mathrm{d}t'\Bigg].\end{aligned} \tag{49}$$

Because we know both the bottom layer bosonic and the fermionic quantities contained in $\boldsymbol{a}$ we can now in principle insert these into the integrals and evaluate the solutions as explicit functions of time. However, since the knowledge of these functions does not provide new insights, we refrain from writing them here.

In summary, the complete solution to the top layer bosonic equation is given by the sum of the homogeneous solutions (46) and (47) and the inhomogeneous solution given by (48) and (49).

7. Discussion

Our aim in this paper has been not only to analyse and solve a fascinating problem in supersymmetric classical mechanics but also to stress the relations between supersymmetric mechanical models in one and more than one dimension. An example is that the fermionic variables should really be thought of as components of a $2n$-dimensional vector Ψ which moves on a $(2n-1)$-sphere. In the one-dimensional case this means that all fermionic vectors move on circles, which is consistent with our findings in [9]. Furthermore, the motion is coupled to the motion of a purely bosonic vector, made of the dynamical variables $\dot{\boldsymbol{x}}$ and $\boldsymbol{U}$, by the supercharges. While these completely determine the fermionic motion in one dimension we have seen that the situation in higher dimensions is more complicated and the supercharges only prescribe a geometric subspace for the true solution. Motion on this subspace is then no longer characterized by the rigid connection of bosonic and fermionic motion found in the one-dimensional case but exhibits a rather interesting time-dependent phase.

We have shown that for rotationally invariant scalar potentials there will always be a conserved angular momentum so that to lowest order bosonic motion takes place in a plane. It is not unreasonable to conjecture that for every direction orthogonal to that plane we will have to solve a one-dimensional problem for the fermionic quantities corresponding to that direction—as was the case for the Coulomb problem. This leads us to another generalization from the one-dimensional case, namely that trivial solutions to the bosonic equations of motion do not imply that the corresponding fermionic equations can only be solved trivially. In fact, for our particular problem we found that the third components of the fermionic quantities still ‘felt’ the presence of the bosonic motion in the plane orthogonal to them.

One of the most fruitful ideas for the higher bosonic layers which can be applied in any dimension turned out to be that solutions for these layers are variations of the solutions of the classical problem. While in one dimension every solution can be conveniently described by the energy and some choice for the initial time, in higher dimensions more of these parameters are needed—in the case of the Coulomb problem we have conveniently related them to the elements describing the classical orbit. So the view taken in [9] for the one-dimensional case, namely that supersymmetric dynamics captures information over a whole range of *energies* of the system, can be confirmed in a more general sense for our higher dimensional model: the bosonic (but Grassmann-valued) part of the solution not only describes one particular classical solution (by its component of order zero) but also includes all possible variations of it with all possible free parameters or, in other words, a bosonic solution corresponds not just to a point in the real parameter space but to something like a fuzzy subset.

For reasons of simplicity we have described in this paper solutions for a Grassmann algebra with two generators. From our experience with the one-dimensional model and the clear links that it exhibits to the three-dimensional system we have studied in this paper, we conjecture that the picture for a larger number of generators is similar to that in the one-dimensional case. There we were able to show that solutions to the fermionic equations corresponding to three generators consist of two parts: one that looks like the first-order solutions themselves (i.e. movement on a circle in one dimension) and one that consists of the *variations* of these solutions with energy and initial time. Similarly, the bosonic equations for four generators are solved by the first and second variations of the lowest order bosonic solutions *and* first variations of the interaction term.

It is, therefore, not unreasonable to assume that the general picture in three dimensions is not much different and that the higher-order equations of motion can be solved by a suitable combination of (first and higher order) variation terms with respect to the free parameters of the physical model in question.

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Appendix. Runge–Lenz vector

In this section we want to outline a proof that the supersymmetric Coulomb problem discussed in this paper does not admit a conserved Runge–Lenz vector. In the classical problem this conserved quantity cannot be constructed as a Noether charge but arises as a consequence of the hidden $SO(3,1)$-symmetry of the problem, explicitly recognizable in our construction of the bottom layer solution to the equations of motion in section 6.1. This symmetry must be broken in the supersymmetric case.

We start with the assumption that there is a conserved Runge–Lenz vector in the supersymmetric problem. Setting all fermionic variables to zero (for our choice of Grassmann algebra this can be accomplished by choosing $Q = \tilde{Q} = P = \tilde{P} = O = \tilde{O} = 0$) we expect to get the classical result for this vector, namely,

$$\boldsymbol{V}_{\mathrm{RL}} = [\boldsymbol{L}, \boldsymbol{p}] - \hat{\boldsymbol{x}} \tag{50}$$

where $\boldsymbol{L}$ is given just by its bosonic part $[\boldsymbol{x}, \boldsymbol{p}]$. (We will use $\boldsymbol{p} \equiv \dot{\boldsymbol{x}}$ for this section.)

Note that the Hamiltonian and the angular momentum vector also have this property. Of course, here all quantities are Grassmann-even rather than real, but this makes no difference and, of course, (50) is a conserved quantity if we set all fermionic terms in the Lagrangian (1) to zero.

Returning now to the general case with non-trivial fermionic variables we take the time-derivative of (50) and find (still taking the classical result for $\boldsymbol{L}$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{V}_{\mathrm{RL}} = -\frac{3}{2}\,\mathrm{i}\frac{2^{\frac{1}{2}}}{|\boldsymbol{x}|^{\frac{3}{2}}}\Bigg[&\boldsymbol{p}\left((\boldsymbol{\psi}_+\cdot\boldsymbol{\psi}_-) - \frac{3}{2}(\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}})\right) + \boldsymbol{\psi}_+(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}})(\boldsymbol{p}\cdot\hat{\boldsymbol{x}}) \\ &- \boldsymbol{\psi}_-(\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})(\boldsymbol{p}\cdot\hat{\boldsymbol{x}}) - \hat{\boldsymbol{x}}\Bigg(2(\boldsymbol{\psi}_+\cdot\boldsymbol{p})(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}}) + 2(\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})(\boldsymbol{\psi}_-\cdot\boldsymbol{p}) \\ &+ (\boldsymbol{\psi}_+\cdot\boldsymbol{\psi}_-)(\boldsymbol{p}\cdot\hat{\boldsymbol{x}}) - \frac{7}{2}(\boldsymbol{\psi}_+\cdot\hat{\boldsymbol{x}})(\boldsymbol{\psi}_-\cdot\hat{\boldsymbol{x}})(\boldsymbol{p}\cdot\hat{\boldsymbol{x}})\Bigg)\Bigg].\end{aligned}$$

The result is clearly non-zero but then we can expect the supersymmetric version of the Runge–Lenz vector to have an additional fermionic piece. However, the derivative of this piece should yield exactly the same result as above with an extra minus sign so that the derivative of the total will be zero and hence $\boldsymbol{V}_{\mathrm{RL}}$ is a conserved quantity. We will now try to construct this fermionic piece.

We start by writing our result for the bosonic part of $\boldsymbol{V}_{\mathrm{RL}}$ in components, so that the linearity in $\boldsymbol{p}$, $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ can be seen in a more explicit way:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V^k_{\mathrm{RL}} = -\frac{3}{2}\,\mathrm{i}\frac{2^{\frac{1}{2}}}{|\boldsymbol{x}|^{\frac{3}{2}}}\Bigg(&\delta^{ij}\delta^{kl} - \frac{3}{2}\delta^{kl}\hat{x}^i\hat{x}^j + \delta^{ik}\hat{x}^j\hat{x}^l + \delta^{jk}\hat{x}^i\hat{x}^l - 2\delta^{il}\hat{x}^j\hat{x}^k - 2\delta^{jl}\hat{x}^i\hat{x}^k \\ &- \delta^{ij}\hat{x}^k\hat{x}^l + \frac{7}{2}\hat{x}^i\hat{x}^j\hat{x}^k\hat{x}^l\Bigg)\psi^i_+\psi^j_-p^l.\end{aligned} \tag{51}$$

We now try to assemble all terms whose derivative could possibly contribute to this result. Fortunately, there are restrictions as to which terms to consider that shorten the list of expressions to look at.

Firstly, we need only to analyse terms quadratic in the fermionic variables: since $\boldsymbol{V}_{\mathrm{RL}}$ is even their number has to be divisible by two, all terms containing no fermionic quantities at all can be assumed to be contained already in (50) and if there were four or more fermionic variables in $\boldsymbol{V}_{\mathrm{RL}}$ the time derivative would still contain the same number of these variables in contradiction to (51). Furthermore, since our system is symmetric under the exchange $\psi_+ \to -\psi_-$, $\psi_- \to \psi_+$ we have to include only terms subject to this symmetry. This leaves us with multiples of the terms

$$\text{(A)} \quad \mathrm{i}\psi_+^i \psi_-^j$$
$$\text{(B)} \quad \mathrm{i}\psi_+^i \psi_+^j + \mathrm{i}\psi_-^i \psi_-^j.$$

When we classify all terms according to the type of their fermionic part and the even or odd number of p^l-variables involved in each one of them, then the time derivative can be seen as acting on these classes as follows:

$$\begin{array}{llcl} & & \dfrac{\mathrm{d}}{\mathrm{d}t} & \\ \text{(I)} & \text{type A, } p^l \text{ even} & \to & \text{type A, } p^l \text{ odd} + \text{type B, } p^l \text{ even} \\ \text{(II)} & \text{type A, } p^l \text{ odd} & \to & \text{type A, } p^l \text{ even} + \text{type B, } p^l \text{ odd} \\ \text{(III)} & \text{type B, } p^l \text{ even} & \to & \text{type A, } p^l \text{ even} + \text{type B, } p^l \text{ odd} \\ \text{(IV)} & \text{type B, } p^l \text{ odd} & \to & \text{type A, } p^l \text{ odd} + \text{type B, } p^l \text{ even.} \end{array}$$

Here p^l even/odd means the number of p^l-variables contained in a term is even or odd, respectively.

Since we want to derive a term of type A with an *odd* number of p^l, namely just one, we can read off from the summary above that we need not consider combinations of type II or III, since their time derivatives do not yield such a term nor can they compensate the B-terms that will arise from combinations I or IV. This means that we only have to consider these latter combinations. However, even those are not unproblematic: as mentioned, they give rise to fermionic terms of type B which we do not want in our derivative, so we must hope that we can cancel these terms against each other.

In the next step we make, preliminarily, the assumption that the number of p^l-variables involved is either zero or one. We will later see that allowing for higher powers of p^l will not change anything.

Up to this point the allowed combinations left are

$$\text{(a)} \quad \mathrm{i}\psi_+^i \psi_-^j \qquad \text{(b)} \quad \left(\mathrm{i}\psi_+^{\mathrm{i}} \psi_+^j + \mathrm{i}\psi_-^{\mathrm{i}} \psi_-^j\right) p^k.$$

We now have to contract these with the remaining natural tensors, namely δ^{ij} and $\hat{x}^i$, to leave only one free index. For terms of type (a) we have the following options: $\hat{x}^i \hat{x}^j \hat{x}^k$, $\hat{x}^i \delta^{jk}$, $\hat{x}^j \delta^{ik}$, $\hat{x}^k \delta^{ij}$.

For terms of type (b) we have some extra restrictions: first, since they are antisymmetric under the exchange $i \leftrightarrow j$ we need not to consider δ^{ij}-terms that contract the two fermionic variables. Second, if we include any $\hat{x}$-term the time-derivative will automatically yield an additional p^l-term so that the result would be quadratic in the p^l—a case which we have excluded above. Therefore the only option left for combination (b) is $\delta^{jk}\delta^{il}$.

We still have to determine the prefactor of each term. It can be derived by dimensional arguments. As can be read off from the Coulomb–Hamiltonian (22) the dimension of energy is length^{-1}. From this we can calculate the dimension of every other dynamical quantity:

$\boldsymbol{x}$	$\boldsymbol{p}$	ψ_+	ψ_-	$\boldsymbol{V}_{\mathrm{RL}}$
length	length$^{-1/2}$	length$^{1/4}$	length$^{1/4}$	1

Since the bosonic part of the Runge–Lenz vector is dimensionless we have to multiply each candidate term for the fermionic part by the appropriate factor of $r \equiv |\boldsymbol{x}|$ to make it dimensionless, too. We end up with the list in the first column of the following two tables. These tables describe the action of the time-derivative on each candidate term. In the top rows we have specified all tensorial combinations that can possibly be generated by applying the time-derivative to the candidate terms in the first column. The table itself then specifies the correct numerical prefactors of each tensor in the time-derivative of a particular candidate term. For example, one can read off from the table entries for the third term that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r^{1/2}}\hat{x}^j\delta^{ik}\mathrm{i}\psi_+^i\psi_-^j\right) = \left(1\,\cdot\,\delta^{ik}\delta^{jl} + \left(-\frac{3}{2}\right)\,\cdot\,\delta^{ik}\hat{x}^j\hat{x}^l\right)r^{-\frac{3}{2}}p^l\mathrm{i}\psi_+^i\psi_-^j + \frac{1}{2}\,\cdot\,\sqrt{2}r^{-2}\delta^{jk}\hat{x}^i\mathrm{i}\psi_+^i\psi_+^j + 1\,\cdot\,\sqrt{2}r^{-2}\delta^{jk}\hat{x}^i\mathrm{i}\psi_-^i\psi_-^j.$$

For comparison we have added the bosonic part of the Runge–Lenz vector and the tensor decomposition of its time-derivative in the last row. The result is the following:

	$r^{-3/2}p^l\mathrm{i}\psi_+^i\psi_-^j$									
	$\delta^{il}\delta^{jk}$	$\delta^{ik}\delta^{jl}$	$\delta^{ij}\delta^{kl}$	$\delta^{il}\hat{x}^j\hat{x}^k$	$\delta^{jl}\hat{x}^i\hat{x}^k$	$\delta^{jk}\hat{x}^i\hat{x}^l$	$\delta^{ik}\hat{x}^j\hat{x}^l$	$\delta^{kl}\hat{x}^i\hat{x}^j$	$\delta^{ij}\hat{x}^k\hat{x}^l$	$\hat{x}^i\hat{x}^j\hat{x}^k\hat{x}^l$
(1) $\frac{1}{r^{1/2}}\hat{x}^i\hat{x}^j\hat{x}^k\mathrm{i}\psi_+^i\psi_-^j$	0	0	0	1	1	0	0	1	0	$-\frac{7}{2}$
(2) $\frac{1}{r^{1/2}}\hat{x}^i\delta^{jk}\mathrm{i}\psi_+^i\psi_-^j$	1	0	0	0	0	$-\frac{3}{2}$	0	0	0	0
(3) $\frac{1}{r^{1/2}}\hat{x}^j\delta^{ik}\mathrm{i}\psi_+^i\psi_-^j$	0	1	0	0	0	0	$-\frac{3}{2}$	0	0	0
(4) $\frac{1}{r^{1/2}}\hat{x}^k\delta^{ij}\mathrm{i}\psi_+^i\psi_-^j$	0	0	1	0	0	0	0	0	$-\frac{3}{2}$	0
(5) $\delta^{il}\delta^{jk}p^l(\mathrm{i}\psi_+^i\psi_+^j + \mathrm{i}\psi_-^i\psi_-^j)$	0	0	0	$\frac{3}{2}\sqrt{2}$	$\frac{3}{2}\sqrt{2}$	$-\frac{3}{2}\sqrt{2}$	$-\frac{3}{2}\sqrt{2}$	0	0	0
$\boldsymbol{V}_{\mathrm{RL,bos}}$	0	0	$-\frac{3}{2}\sqrt{2}$	$3\sqrt{2}$	$3\sqrt{2}$	$-\frac{3}{2}\sqrt{2}$	$-\frac{3}{2}\sqrt{2}$	$\frac{9}{4}\sqrt{2}$	$\frac{3}{2}\sqrt{2}$	$-\frac{21}{4}\sqrt{2}$

	$\sqrt{2}r^{-2}\delta^{jk}\hat{x}^i\mathrm{i}\psi_+^i\psi_+^j$	$\sqrt{2}r^{-2}\delta^{jk}\hat{x}^i\,\mathrm{i}\psi_-^i\psi_-^j$
(1) $\frac{1}{r^{1/2}}\hat{x}^i\hat{x}^j\hat{x}^k\,\mathrm{i}\psi_+^i\psi_-^j$	0	0
(2) $\frac{1}{r^{1/2}}\hat{x}^i\delta^{jk}\,\mathrm{i}\psi_+^i\psi_-^j$	1	$\frac{1}{2}$
(3) $\frac{1}{r^{1/2}}\hat{x}^j\delta^{ik}\,\mathrm{i}\psi_+^i\psi_-^j$	$\frac{1}{2}$	1
(4) $\frac{1}{r^{1/2}}\hat{x}^k\delta^{ij}\mathrm{i}\psi_+^i\psi_-^j$	0	0
(5) $\delta^{il}\delta^{jk}p^l\,(\mathrm{i}\psi_+^i\psi_+^j + \mathrm{i}\psi_-^i\psi_-^j)$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$
$\boldsymbol{V}_{\mathrm{RL,bos}}$	0	0

We now have to choose appropriate coefficients for the candidate terms such that their sum equals the bosonic part of $\boldsymbol{V}_{\mathrm{RL}}$ (with an extra minus sign). Looking at the coefficients of $\delta^{il}\delta^{jk}$ and $\delta^{ik}\delta^{jl}$ we can immediately see that terms (2) and (3) cannot occur since these tensorial combinations are not part of the bosonic Runge–Lenz vector and cannot be compensated by any other term. Looking then at the second table we can conclude that term (5) cannot occur either—since it contributes a non-zero factor to both columns whereas $\boldsymbol{V}_{\mathrm{RL}}$ does not (and the terms (2) and (3) have to be zero as deduced above).

The column for $\delta^{ij}\delta^{kl}$ tells us that the coefficient of the fourth term has to be $\frac{3}{2}\sqrt{2}$—in contradiction to the column for $\delta^{ij}\hat{x}^k\hat{x}^l$ which says that the same coefficient has to be $\sqrt{2}$. Similar contradictions arise for our first candidate term when we compare the columns for $\delta^{il}\hat{x}^j\hat{x}^k$, $\delta^{jl}\hat{x}^i\hat{x}^k$, $\delta^{kl}\hat{x}^i\hat{x}^j$ and $\hat{x}^i\hat{x}^j\hat{x}^k\hat{x}^l$. This leaves us no other option than to conclude that none of our candidate terms can be added to the bosonic part of the Runge–Lenz vector to achieve a zero derivative.

One loophole remains. We have excluded further terms of type (b) on the grounds that they yield terms quadratic in momentum. We might hope, however, that in the right combination these quadratic terms can cancel each other. There are two possibilities, which can be found in the first column of the following table; all other tensorial combinations either yield zero or are up to minus sign identical to the two choices below via the index exchange $i \leftrightarrow j$. On taking the time derivative of both terms the crucial issue is whether the terms quadratic in p^l can be cancelled against each other, so only quadratic components are shown here:

	$\frac{1}{r}p^l p^m(\mathrm{i}\psi_+^i\psi_+^j + \mathrm{i}\psi_-^i\psi_-^j)$				
	$\hat{x}^i\delta^{jm}\delta^{kl}$	$\hat{x}^i\delta^{jk}\delta^{lm}$	$\hat{x}^l\delta^{im}\delta^{jk}$	$\hat{x}^i\hat{x}^k\hat{x}^l\delta^{jm}$	$\hat{x}^i\hat{x}^l\hat{x}^m\delta^{jk}$
$\hat{x}^i\hat{x}^k\delta^{jl}p^l(\mathrm{i}\psi_+^i\psi_+^j + \mathrm{i}\psi_-^i\psi_-^j)$	1	0	0	−2	0
$\hat{x}^i\hat{x}^l\delta^{jk}p^l(\mathrm{i}\psi_+^i\psi_+^j + \mathrm{i}\psi_-^i\psi_-^j)$	0	1	1	0	−2

Evidently, the quadratic terms do not cancel each other. We conclude that no term at most linear in momentum exists, which could be added to $\boldsymbol{V}_{\mathrm{RL,bos}}$ to give a conserved quantity in the supersymmetric model.

We can of course ask the question whether the fermionic part of the Runge–Lenz vector could be quadratic or of some higher power in p^k. We will now show that the answer to this question is no. The reason for this can be understood by looking at a general term such as

$$A^{ijk_1\cdots k_m l} f(r) p^{k_1}\cdots p^{k_m}\mathrm{i}\psi_+^i\psi_-^j \tag{52}$$

where $A^{ijk_1\cdots k_m l}$ (l is a free index) is a tensor constructed from two types of building blocks, δ^{pq} and $\hat{x}^p$, and $f(r)$ is some well-determined power of r. Since the tensor is of type A we can infer that the number m of momentum variables has to be even. Furthermore, to keep (52) dimensionless $f(r)$ has to be $r^{\frac{1}{2}(m-1)}$. Clearly, for every $m \geqslant 2\, f(r)$ is a non-trivial function of r. This in turn means that the time-derivative in acting on $f(r)$ will produce an extra p^k variable.

There are only two ways to solve this problem: either all the terms in the time derivative of (52) containing extra momentum variables cancel each other—or we have to make use of yet higher powers of p^k in $\boldsymbol{V}_{\mathrm{RL,ferm}}$ to compensate. However, those very terms will by the same principle give rise to another extra power of p^k in the derivative and so on. For this series to stop at some point we must hope that all terms in the derivative of $A^{ijk_1\cdots k_m l}$ with $m+1$ momentum variables cancel each other. We find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(A^{ijk_1\cdots k_m l} r^{\frac{1}{2}(m-1)} p^{k_1}\cdots p^{k_m}\,\mathrm{i}\psi_+^i\psi_-^j\big) = \frac{\mathrm{d}}{\mathrm{d}t}(A^{ijk_1\cdots k_m l}) r^{\frac{1}{2}(m-1)} p^{k_1}\cdots p^{k_m}\,\mathrm{i}\psi_+^i\psi_-^j$$
$$+ A^{ijk_1\cdots k_m l}\frac{1}{2}(m-1) r^{\frac{1}{2}(m-3)}\hat{x}^{k_{m+1}} p^{k_{m+1}} p^{k_1}\cdots p^{k_m}\,\mathrm{i}\psi_+^i\psi_-^j + \cdots$$

where the dots indicate further terms containing less than $m+1$ momentum variables, in which we are not interested here.

As mentioned above $A^{ijk_1\cdots k_m l}$ consists of a sum of tensors built from δ^{pq} and $\hat{x}^p$ terms. While the time derivative does not act on the first type of term, it acts on the second:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}^p = \frac{1}{r}(\delta^{pl} - \hat{x}^p\hat{x}^l)p^l.$$

Thus we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{ijk_1\cdots k_m l} = \frac{1}{r}\tilde{A}^{ijk_1\cdots k_m k_{m+1} l}p^{k_{m+1}}$$

where $\tilde{A}^{ijk_1\cdots k_m k_{m+1} l}$ is a new tensor derived from $A^{ijk_1\cdots k_m l}$ in the following way, which can be seen as an algebraic version of the Leibniz rule: we replace one $\hat{x}^p$ tensor in $A^{ijk_1\cdots k_m l}$ with $(\delta^{pk_{m+1}} - \hat{x}^p\hat{x}^{k_{m+1}})$, repeat this step for all other $\hat{x}^p$ tensors contained in $A^{ijk_1\cdots k_m l}$—replacing always one tensor at a time—and then sum up every contribution. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{ijk_1\cdots k_m l}r^{\frac{1}{2}(m-1)}p^{k_1}\cdots p^{k_m}\mathrm{i}\psi_+^i\psi_-^j\right) = \left(\tilde{A}^{ijk_1\cdots k_m k_{m+1} l} + \frac{1}{2}(m-1)A^{ijk_1\cdots k_m l}\hat{x}^{k_{m+1}}\right)$$
$$\times r^{\frac{1}{2}(m-3)}p^{k_1}\cdots p^{k_m}p^{k_{m+1}}\mathrm{i}\psi_+^i\psi_-^j + \cdots.$$

So, if we want a cancellation of all terms containing $m+1$ momentum variables, we need

$$\tilde{A}^{ijk_1\cdots k_m k_{m+1} l} + \tfrac{1}{2}(m-1)A^{ijk_1\cdots k_m l}\hat{x}^{k_{m+1}} = 0. \tag{53}$$

It remains to be shown that this equation cannot be satisfied. Starting with

$$A_0^{ijk_1\cdots k_m l} = \hat{x}^i\hat{x}^j\hat{x}^{k_1}\cdots\hat{x}^{k_m}\hat{x}^l \tag{54}$$

we find

$$\tilde{A}_0^{ijk_1\cdots k_m k_{m+1} l} = -(m+3)\hat{x}^i\hat{x}^j\hat{x}^{k_1}\cdots\hat{x}^{k_m}\hat{x}^{k_{m+1}}\hat{x}^l + \delta^{ik_{m+1}}\hat{x}^j\hat{x}^{k_1}\cdots\hat{x}^{k_m}\hat{x}^l$$
$$+\cdots+\hat{x}^i\hat{x}^j\hat{x}^{k_1}\cdots\hat{x}^{k_m}\delta^{lk_{m+1}}.$$

Inserting these results into (53) we find the necessary condition $-\frac{1}{2}(m+7) = 0$, i.e. $m = -7$, which is a clear contradiction, so a term such as (54) cannot occur. Replacing two $\hat{x}^k$ terms by a δ-tensor we arrive at

$$A_1^{ijk_1\cdots k_m l} = \hat{x}^i\hat{x}^j\hat{x}^{k_1}\cdots\hat{x}^{k_{m-1}}\delta^{k_m l} \tag{55}$$

to name but one possible choice. The corresponding $\tilde{A}$ is then given by

$$\tilde{A}_1^{ijk_1\cdots k_m k_{m+1} l} = -(m+1)\hat{x}^i\hat{x}^j\hat{x}^{k_1}\cdots\hat{x}^{k_{m-1}}\delta^{k_m l}\hat{x}^{k_{m+1}} + \cdots$$

where the dots indicate further terms which contain less than $m+1$ $\hat{x}^k$-tensors. Equation (53) then yields $-\frac{1}{2}(m+3) = 0$, which is again a contradiction.

One can now see that as we subsequently replace two $\hat{x}^k$-terms by one δ-tensor, equation (53) gives that $-\frac{1}{2}(m+7-2p) = 0$, where p is the total number of $\hat{x}^k$-terms exchanged as compared to (54). But this shows that m has to be an odd integer in contradiction to our demand that m be an even number. As a result no type A tensor can have the maximal number of momentum variables in the fermionic part of the Runge–Lenz vector.

We refrain from repeating the argument for type B tensors since the result is the same: the time-derivative in acting on these tensors increases the number of p^k-variables by one and there is no way of cancelling all these extra unwanted terms against each other. We can therefore safely conclude that even the introduction of higher orders of momentum does not change the overall result, namely that a supersymmetric version of the Runge–Lenz vector does not exist.

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